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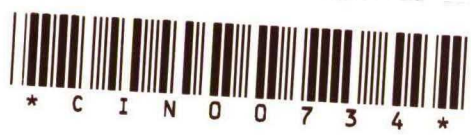
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**The Consistency Principle For Games
In Strategic Form**

**by Bezalel Peleg and
Stef Tijs**

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THE CONSISTENCY PRINCIPLE FOR GAMES IN STRATEGIC FORM*

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Abstract. We start with giving an axiomatic characterization of the Nash equilibrium (NE) correspondence in terms of consistency, converse consistency, and one-person rationality. Then axiomatizations are given of the strong NE correspondence, the coalition-proof NE correspondence and the semi-strong NE. In all these characterizations consistency and suitable variants of converse consistency play a role. Finally, the dominant NE correspondence is characterized. We also indicate how to generalize our results to Bayesian and extensive games.

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1. Introduction

The consistency property of solutions of cooperative games is well-known. For recent formulations the reader is referred to Maschler [1990] and Thomson [1990]. Harsanyi [1959] was the first to use consistency in extending Nash's solution of two-person bargaining problems to n -person problems. Lensberg [1988] characterized the Nash solution of the bargaining problem by using consistency and three other standard properties. Additional applications of the consistency principle to the theory of bargaining are collected in Thomson and Lensberg [1989]. Recently, Peters, Tijs, and Zarzuelo [1991] characterized the Kalai-Smorodinsky solution by consistency.

Davis and Maschler [1965] started the investigation of consistency of the kernel. A closely related solution, the prekernel, was axiomatized by Peleg [1986] by using consistency and converse consistency. Also, consistency plays a central role in the axiomatizations of the prenucleolus (Sobolev [1975]), the nucleolus (Potters [1991] and Snijders [1991]), the generalized nucleolus (Maschler, Potters, and Tijs [1992]), the core (Peleg [1986], [1985], and Tadenuma [1992]), the τ -value (Driessen [1992]), and the Shapley value (Hart and Mas-Colell [1989]).

The reader is now referred to Thomson's [1990] comprehensive survey of the consistency principle, for further applications of consistency to bankruptcy and taxation problems, quasi-linear cost allocation problems, and resource allocation problems. Driessen's survey [1991] is also very instructive.

We now verbally describe the consistency principle for games in strategic form. If G is a game, S is a subset of the set of players of G , and x is a strategy profile for the grand coalition, then the *reduced game* $G^{S,x}$ is the game faced by the members of S , when the members of $N \setminus S$ leave the game after choosing $x_{N \setminus S}$ (see Section 2 for the precise definition). A *solution* φ on a set Γ of games assigns for every game in Γ a set of strategy profiles. φ is *consistent* if for every $G \in \Gamma$, a coalition S of the players in G , and $x \in \varphi(G)$, the restriction of x to S , x_S , is in $\varphi(G^{S,x})$. The foregoing definitions have already appeared in Aumann [1987] and, less explicitly, in Bernheim, Peleg, and Whinston [1987]. However,

as far as we know, this paper is the first systematic study of the consistency principle for solutions of games in strategic form.

We now review briefly the contents of the paper. Section 2 contains two axiomatizations of the Nash correspondence (see Theorem 2.12 and Corollary 2.22). The main result, Theorem 2.12, characterizes the Nash Equilibrium as the unique solution that satisfies one-person rationality, consistency, and converse consistency. The three basic concepts: reduced game, consistency, and converse consistency are defined in the beginning of the section. The strong Nash equilibrium is axiomatized in Section 3 (see Theorem 3.2 and Corollary 3.9). We also characterize the strictly strong Nash equilibria (see Theorem 3.7). Coalition-proof Nash equilibria and semi-strong Nash equilibria are characterized in Section 4. Section 5 is devoted to an axiomatization of dominant strategies (see Theorem 5.6). In Sections 6 and 7 we indicate how to generalize our results to Bayesian games and extensive games respectively. Section 7 also contains an axiomatization of the subgame perfect equilibria of games with perfect information. Concluding remarks and open problems appear in Section 8.

2. Axiomatic Characterizations of the Nash Equilibrium Correspondence

In this section we introduce some properties of solutions of games in strategic form that are satisfied by the Nash correspondence. The first three properties are used to axiomatize the set of Nash equilibria. Additional properties will be defined in subsequent sections in order to characterize some refinements of the Nash equilibrium.

A game in strategic form is a system $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, where N is a finite set of players; A_i , $i \in N$, is the (non-empty) set of strategies of i ; and $u_i : \prod_{j \in N} A_j \rightarrow R$ is the payoff function of player $i \in N$. (Here R denotes the set of real numbers.) Let $\emptyset \neq S \subseteq N$. We denote $A_S = \prod_{i \in S} A_i$. Also, we denote $A = A_N$. Let Γ be a set of games. A solution on Γ is a function φ that assigns to each game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ a subset $\varphi(G)$ of A .

Let Γ be a set of games and let φ be a solution on Γ .

Definition 2.1. φ satisfies *one-person rationality* (OPR) if for every one-person game $G = (\{i\}, A_i, u_i)$ in Γ

$$\varphi(G) = \{x_i \in A_i \mid u_i(x_i) \geq u_i(y_i) \text{ for all } y_i \in A_i\}.$$

OPR is a consequence of the rationality of the players. The Nash correspondence and all its refinements satisfy OPR.

Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game, let $\emptyset \neq S \subseteq N$, and let $x \in A$. The *reduced game* of G with respect to (w.r.t.) S and x is the game $G^{S,x} = (S, (A_i)_{i \in S}, (u_i^x)_{i \in S})$ where $u_i^x(y_S) = u_i(y_S, x_{N \setminus S})$ for all $y_S \in A_S$ and $i \in S$. Our definition of reduced games is simple and has a straightforward interpretation. Let $\emptyset \neq S \subseteq N$ and $x \in A$. If it is common knowledge among the members of S that the members of $N \setminus S$ have chosen the strategies x_i , $i \in N \setminus S$, then the members of S are faced with the game $G^{S,x}$.

We remark that the “usual” definitions of reduced games of cooperative games are more complicated (see, e.g., Davis and Maschler [1965] and Hart and Mas-Colell [1989]).

Now we shall define consistency of solutions of games in strategic form. A family Γ of games is *closed* if $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $\emptyset \neq S \subseteq N$ and $x \in A$ imply that $G^{S,x} \in \Gamma$.

Definition 2.2. Let Γ be a closed family of games and let φ be a solution on Γ . φ is *consistent* (CONS) if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ in Γ , $\emptyset \neq S \subseteq N$, and $x \in \varphi(G)$, $x_S \in \varphi(G^{S,x})$.

Consistency of solutions of cooperative games has been extensively investigated by many authors (see the survey papers of Thomson [1990] and Driessen [1991]). Also, the reader may find lucid explanations of the consistency principle for cooperative games in both Maschler [1990] and Thomson [1990]. For games in strategic form consistency has a simple interpretation. If $G \in \Gamma$ and $x \in \varphi(G)$, then for every $\emptyset \neq S \subseteq N$, x_S is also prescribed by φ to the game G restricted to S , that is, to the game $G^{S,x}$. Equivalently, if the members of S know that the members of $N \setminus S$ have chosen $x_{N \setminus S}$ and left the game G , then they do not have to revise their strategies.

Although, as far as we know, consistency of solutions of games in strategic form is systematically discussed here for the first time, in some definitions of such solutions the role of consistency is implicit, or even explicit. For example, the “main part” of the definition of Nash equilibria is the requirement that it will be “consistent” with its prescription to one-person reduced games. However, the Nash equilibrium (NE) clearly satisfies the stronger consistency property of Definition 2.2. Reduced games and consistency appear explicitly in Aumann [1987] and in the definition of coalition-proof Nash equilibria (CPNE) (see Bernheim, Peleg, and Whinston [1987]). However, the foregoing concepts were not further studied in those papers.

Remark 2.3. Let Γ be a closed family of games and let φ be a solution on Γ . The following condition implies the consistency of φ : For every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ in Γ , $i \in N$, and $x \in \varphi(G)$, $x_{N \setminus \{i\}} \in \varphi(G^{N \setminus \{i\}, x})$. The proof of this remark is left to the reader.

Some refinements of NE do not satisfy consistency. This is shown by the following example due to E. van Damme.

Example 2.4. Let the three-person game G_0 be given by the following pair of matrices:

	L	R		L	R
T	1, 1, 1	1, 0, 1	T	0, 1, 0	0, 0, 0
B	1, 1, 1	0, 0, 1	B	1, 1, 0	0, 0, 0
	D			U	

Let Γ consist of the mixed extension G_* of G_0 and all its reduced games. Then Γ is a closed family of games. Further, for each $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ in Γ let

$$\text{PERF}(G) = \{x \in A \mid x \text{ is a perfect equilibrium of } G\}$$

(see Selten [1975] and Myerson [1978]). We note that $x = (B, L, D) \in \text{PERF}(G_*)$, but $(B, L) \notin \text{PERF}(G_*^{(1,2), x})$. Thus, PERF is not consistent. This example also shows that

the correspondences of proper equilibria (Myerson [1978]) and stable equilibria (Kohlberg and Mertens [1986]) do not satisfy CONS.

Now we proceed to introduce converse consistency of solutions of games in strategic form. Let Γ be a closed family of games and let φ be a solution on Γ . If $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is in Γ and $|N| \geq 2$, then we denote

$$(2.1) \quad \tilde{\varphi}(G) = \{x \in A \mid \text{for every } S \subset N, S \neq \emptyset, N, x_S \in \varphi(G^{S,z})\}$$

(if D is a finite set, then $|D|$ denotes the number of members of D).

Definition 2.5. A solution φ on a closed family of games Γ satisfies *converse consistency* (COCONS) if for every $G \in \Gamma$ with at least two players, $\tilde{\varphi}(G) \subset \varphi(G)$.

We remark that consistency can be defined by the reverse inclusion, namely $\varphi(G) \subset \tilde{\varphi}(G)$ for every $G \in \Gamma$. This explains our terminology. Converse consistency of solutions of cooperative games was defined in Peleg [1986]. However, it was first used in Harsanyi [1959]. Converse consistency may produce iterative algorithms that converge to solution points (see Thomson [1991] for a recent investigation and survey). Clearly, the NE solution satisfies COCONS. In fact, it satisfies the following stronger property.

Definition 2.6. Let Γ be a closed set of games and let φ be a solution on Γ . φ satisfies COCONS₀ if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ in Γ and $x \in A$ it is true that

$$(2.2) \quad [\cup\{S \subset N \mid S \neq \emptyset \text{ and } x_S \in \varphi(G^{S,z})\} = N] \Rightarrow x \in \varphi(G)$$

We summarize our remarks on the Nash correspondence by the following proposition:

PROPOSITION 2.7. Let Γ be a closed family of games. The NE solution on Γ satisfies OPR, CONS, and COCONS.

In fact, OPR, CONS, and COCONS characterize the NE. This is shown by the following two propositions.

PROPOSITION 2.8. Let φ be a solution on a closed family of games Γ . If φ satisfies OPR and CONS, then $\varphi(G) \subset NE(G)$ for every $G \in \Gamma$.

PROOF: Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ and $x \in \varphi(G)$. By CONS, $x_i \in \varphi(G^{(i), x})$ for each $i \in N$. By OPR $u_i^x(x_i) \geq u_i^x(y_i)$ for all $y_i \in A_i$ and $i \in N$. Thus

$$u_i(x_i, x_{N \setminus \{i\}}) \geq u_i(y_i, x_{N \setminus \{i\}}) \text{ for all } y_i \in A_i \text{ and } i \in N.$$

Hence, x is an NE of G . Q.E.D.

Remark 2.9. By Proposition 2.8, every solution φ that satisfies OPR and CONS on a closed family of games Γ is a refinement of the NE, that is $\varphi(G) \subset NE(G)$ for all $G \in \Gamma$. The converse claim is not true (see Example 2.4).

PROPOSITION 2.10. *Let Γ be a closed family of games and let φ be a solution on Γ . If φ satisfies OPR and COCONS, then $\varphi(G) \supset NE(G)$ for every $G \in \Gamma$.*

PROOF: We prove the foregoing inclusion by induction on the number of players. Let $G \in \Gamma$ be a one-person game. Then $\varphi(G) \supset NE(G)$ by OPR. Now assume that $NE(G) \subset \varphi(G)$ for all t -person games in Γ where $t \leq k$ and $k \geq 1$. Let $G_0 \in \Gamma$ be a $(k+1)$ -person game. Because the Nash solution satisfies CONS, $NE(G) \subset \widetilde{NE}(G)$ (see (2.1)). By the induction hypothesis $\widetilde{NE}(G) \subset \tilde{\varphi}(G)$, and by COCONS, $\tilde{\varphi}(G) \subset \varphi(G)$. Hence, $NE(G) \subset \varphi(G)$. Q.E.D.

Remark 2.11. Proposition 2.10 implies that every strict refinement of the NE solution that satisfies OPR does *not* satisfy COCONS.

Combining Propositions 2.7, 2.8, and 2.10 we obtain the following characterization of the NE correspondence.

THEOREM 2.12. *A solution φ on a closed family of games Γ satisfies OPR, CONS, and COCONS, if and only if $\varphi = NE$ (i.e., $\varphi(G) = NE(G)$ for every $G \in \Gamma$).*

COROLLARY 2.13. *Let G be a game in strategic form and let $\Gamma(G)$ be the minimal closed family that contains G (i.e., $\Gamma(G)$ consists of G and all its reduced games). If a solution φ on $\Gamma(G)$ satisfies OPR, CONS, and COCONS, then $\varphi(G) = NE(G)$.*

Corollary 2.13 follows from the proof of Theorem 2.12. See Hart and Mas-Colell [1989] for a similar result on a minimal domain for which there is an axiomatization of the Shapley value. See also Neyman [1989].

Remark 2.14. It is possible to replace COCONS by COCONS₀ in Theorem 2.12 (see (2.2)). The proof is left to the reader.

We shall now prove that the three axioms which characterize the NE are logically independent. For this purpose we consider the following families of games. Let $I = \{1, 2, 3, \dots\}$ be the set of natural numbers. Denote

(2.3)

$$P = \{G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \mid \emptyset \neq N \subset I, |N| < \infty \text{ and } |A_i| < \infty \text{ for all } i \in N\}.$$

P is the (closed) set of all finite games. Now let M be the (closed) set of all mixed extensions of games in P . We shall not deal with the question when a closed family of games Γ is rich enough so that our three axioms are independent on Γ .

Example 2.15. Let φ_1 be defined on P (see (2.3))

$$\varphi_1(N, (A_i)_{i \in N}, (u_i)_{i \in N}) = \prod_{i \in N} A_i.$$

Then φ_1 satisfies CONS and COCONS but not OPR.

Example 2.16. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in M$. For each $i \in N$ let $v_i(G)$ be the minmax payoff of i in G , that is

$$(2.4) \quad v_i(G) = \min_{x_{N \setminus \{i\}} \in A_{N \setminus \{i\}}} \max_{x_i \in A_i} u_i(x_i, x_{N \setminus \{i\}}).$$

Clearly, for every $\emptyset \neq S \subset N$ and $x \in A$, $v_i(G^{S,x}) \geq v_i(G)$ for all $i \in S$. Now define

$$\varphi_2(G) = \{x \in A \mid u_i(x) \geq v_i(G) \text{ for every } i \in N\}.$$

Then φ_2 satisfies OPR and COCONS on M . Because $\varphi_2 \neq NE$, φ_2 does not satisfy CONS.

Example 2.17. For each $G \in M$ let $SNE(G)$ be the set of strong Nash equilibria of G (see Section 3). Clearly, SNE satisfies OPR and CONS but not COCONS.

Remark 2.18. The reader might ask what we achieved by Theorem 2.12. In order to reply, let us notice that the definition of the NE is cyclic. Indeed, let $G = (\{1, 2\}; A_1, A_2, u_1, u_2)$ be a two-person game and let $x \in A$. Then x is a NE iff the following conditions are satisfied: (1) If 2 chooses x_2 , then 1 may choose x_1 because x_1 is a best reply to x_2 ; (2) 2 may choose x_2 if 1 chooses x_1 , because x_2 is a best reply to x_1 . By Theorem 2.12 we decompose this (cyclic) definition into three (independent) properties which have straightforward formulations and are intuitively acceptable.

The NE has well-known additional properties which do not appear in our axiomatizations. For example, it is independent of the names of the players (anonymity), and it is invariant under permissible transformations of the payoff functions. We are now going to discuss two additional properties of the NE solution.

Definition 2.19. Let Γ be a set of games and let φ be a solution on Γ . φ satisfies *independence of irrelevant strategies* (IIS) if the following claim is true. If $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $x \in \varphi(G)$, $x_i \in B_i \subseteq A_i$ for all $i \in N$, and $G^* = (N, (B_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, then $x \in \varphi(G^*)$.

Clearly, the NE solution satisfies IIS. In order to find the relationship between IIS and our previous axioms we need one more axiom. Also, we note that if $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is a game, $d \in N$, and $|A_d| = 1$, then d is a *dummy* in G .

Definition 2.20. Let Γ be a closed family of games and let φ be a solution on Γ . φ satisfies the *dummy axiom* (DUM) if for every game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ and every dummy d in G , $\varphi(G) = A_d \times \varphi(G^{N \setminus \{d\}}, x)$, where x may be any member of A .

The dummy axiom needs no explanation. Clearly the NE correspondence satisfies DUM. A family Γ of games is *closed** if for every game $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ and $\emptyset \neq B_i \subseteq A_i$, $i \in N$, the game $G^* = (N, (B_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$. Γ is *closed²* if it is both closed and closed*. The family P (see (2.3)) is closed². Also, the set of ordinal potential games is closed² (see Monderer and Shapley [1992]).

PROPOSITION 2.21. *Let Γ be a closed² family of games and let φ be a solution on Γ . If φ satisfies IIS and DUM, then φ also satisfies CONS.*

PROOF: Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $x \in \varphi(G)$, and $\emptyset \neq S \subseteq N$. By IIS

$$x \in \varphi(N, \{x_i\}_{i \notin S}, (A_i)_{i \in S}, (u_i)_{i \in N}).$$

By DUM $x \in \{x_{N \setminus S}\} \times \varphi(G^{S, x})$. Hence $x_S \in \varphi(G^{S, x})$. Q.E.D.

COROLLARY 2.22. *Let Γ be a closed² family of games. The NE correspondence is the unique solution on Γ that satisfies OPR, COCONS, IIS, and DUM.*

Corollary 2.22 follows from Theorem 2.12 and Proposition 2.21.

Remark 2.23. The axioms OPR, COCONS, IIS, and DUM are logically independent. Indeed, Example 2.15 satisfies COCONS, IIS, and DUM, but not OPR. Example 2.17 satisfies OPR, IIS, and DUM but not COCONS. Example 2.16 satisfies OPR, COCONS, and DUM but not IIS. Finally, the following example shows that DUM is independent of OPR, COCONS, and IIS.

Example 2.24. Let the three-person game G_0 (in pure strategies) be given by the following matrix

	<i>L</i>	<i>R</i>
<i>T</i>	1, -1, 0	-1, 1, 0
<i>B</i>	-1, 1, 0	1, -1, 0
	<i>D</i>	

Furthermore, let $\Gamma = \Gamma(G_0)$ be the minimal closed² family that contains G_0 . We define a solution φ on Γ by the following rule. If $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is in Γ and $|N| \leq 2$, then $\varphi(G) = NE(G)$; and if $|N| = 3$ then $\varphi(G) = A$. As the reader may verify, φ satisfies OPR, COCONS, and IIS, but not DUM.

3. Axiomatization of the Strong Nash Equilibrium

Strong Nash equilibria were defined in Aumann [1959]. In this section we shall give two axiomatizations of strong equilibria. First we recall some definitions.

Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form. $x \in A$ is a *strong Nash equilibrium* (SNE) if for every $\emptyset \neq S \subseteq N$ and every $y_S \in A_S$ there exists $i \in S$ such that $u_i(x) \geq u_i(y_S, x_{N \setminus S})$. $x \in A$ is *weakly Pareto-optimal* (WPO) if for every $y \in A$ there exists $i \in N$ such that $u_i(x) \geq u_i(y)$. Finally, $x \in A$ is *Pareto-optimal* (PO) if there is no $y \in A$ such that $u_i(y) \geq u_i(x)$ for all $i \in N$ and $u_j(y) > u_j(x)$ for at least one $j \in N$.

Let G be a game. We denote by $SNE(G)$ the set of strong Nash equilibria of G . As the reader may verify, the solution SNE satisfies OPR and WPO. Also, on closed domains it satisfies CONS but not COCONS. We shall now formulate a weaker notion of converse consistency which the SNE satisfies.

Definition 3.1. Let φ be a solution on a closed family of games Γ . φ satisfies $COCONS_3$ if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ with $|N| \geq 2$ and $x \in A$ it is true that

$$(3.1) \quad [x \in \tilde{\varphi}(G) \text{ and } x \text{ is WPO}] \Rightarrow x \in \varphi(G)$$

(see (2.1)).

THEOREM 3.2. Let Γ be a closed family of games. There is a unique solution on Γ that satisfies CONS, $COCONS_3$, and WPO and it is the SNE.

PROOF: As the reader may verify, SNE satisfies CONS, $COCONS_3$, and WPO. Now let φ be a solution on Γ that satisfies the foregoing three axioms. We prove by induction on the number of players that $\varphi(G) = SNE(G)$ for every $G \in \Gamma$. Clearly, WPO implies OPR. Therefore $\varphi(G) = SNE(G)$ if G is a one-person game. Now assume that $SNE(G) = \varphi(G)$ for every m -person game $G \in \Gamma$, where $1 \leq m \leq k$ and $k \geq 1$, and let $G_0 \in \Gamma$ be a $(k+1)$ -person game. Further, let $x \in \varphi(G_0)$. By CONS, $x \in \tilde{\varphi}(G_0)$, and by the induction hypothesis $\tilde{\varphi}(G_0) = \widetilde{SNE}(G_0)$. Hence, by WPO (of φ) and $COCONS_3$ (of SNE), $x \in SNE(G_0)$. Thus $\varphi(G_0) \subset SNE(G_0)$. Similarly, we may prove that $SNE(G_0) \subset \varphi(G_0)$. Q.E.D.

The proof of Theorem 3.2 implies the following corollary.

COROLLARY 3.3. *Let Γ be a closed family of games. If φ is a solution on Γ that satisfies CONS and WPO, then $\varphi(G) \subset SNE(G)$ for every $G \in \Gamma$.*

We shall now show that the three axioms in Theorem 3.2 are independent. Indeed, WPO, that is the correspondence that assigns to each game its set of weakly Pareto-optimal strategy profiles, satisfies on M (see Section 2 where M is defined) COCONS₃ and WPO but not CONS. Also, Example 2.15 satisfies CONS and COCONS₃ but not WPO. The next two examples complete the proof of independence.

Example 3.4. Let $DS(G)$ be the set dominant strategies of G for each $G \in M$ (see Section 5). Then DS satisfies CONS and WPO but not COCONS₃.

For the next example we need the following definition.

Definition 3.5. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game. $x \in A$ is a *strictly strong Nash equilibrium* (SSNE) if if there do not exist a coalition S and $y_S \in A_S$, such that $u_i(y_S, x_{N \setminus S}) \geq u_i(x)$ for all $i \in S$, and for at least one $j \in S$ $u_j(y_S, x_{N \setminus S}) > u_j(x)$.

The set of strictly strong Nash equilibria of a game G is denoted by $SSNE(G)$. SSNE satisfies CONS and WPO but not COCONS₃. In order to characterize the SSNE we need the following axiom.

Definition 3.6. Let φ be a solution on closed family of games Γ . φ satisfies COCONS₄ if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ with $|N| \geq 2$ and $x \in A$ it is true that

$$(3.2) \quad [x \in \tilde{\varphi}(G) \text{ and } x \text{ is PO}] \Rightarrow x \in \varphi(G)$$

(see (2.1)).

THEOREM 3.7. *Let Γ be a closed family of games. There is a unique solution on Γ that satisfies CONS, COCONS₄, and PO and it is the SSNE. Furthermore, CONS, COCONS₄, and PO are independent.*

SSNE is used in Borm and Tijs [1992]. The proof of Theorem 3.7 is left to the reader. The reader may also formulate the analog of Corollary 3.3 for the SSNE solution.

Remark 3.8. Let G be a game. If $x \in SNE(G)$ then $\underline{u}(x) = (u_1(x), \dots, u_n(x))$ belongs to the β -core of G (see, e.g., Lemma 2.36 of Moulin and Peleg [1982] for a proof of a similar result). Similarly, the solution SSNE is related to the strong β -core of G (see Borm and Tijs [1992]). On the role of the strong core in exchange markets see Wako [1991a,b].

We now observe that SNE satisfies IIS (see Definition 2.19). Thus, combining proposition 2.21 and Theorem 3.2 we obtain our second characterization of the SNE solution.

COROLLARY 3.9. *Let Γ be a closed² family of games (see Section 2). There is a unique solution on Γ that satisfies IIS, WPO, COCONS₃, and DUM, and it is the SNE.*

4. Characterizations of Coalition-Proof Nash Equilibria and Semi-Strong Equilibria

In this section we shall axiomatize two additional refinements of the NE. We now introduce the first. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game, let $x \in N$ and let $\emptyset \neq S \subseteq N$. An *internally consistent improvement* (ICI) of S upon x is defined by induction on $|S|$. If $|S| = 1$, that is $S = \{i\}$ for some $i \in N$, then $y_i \in A_i$ is an ICI of i upon x if it is an improvement upon x , that is, $u_i(y_i, x_{N \setminus \{i\}}) > u_i(x)$. If $|S| > 1$ then $y_S \in A_S$ is an ICI of S upon x if (i) $u_i(y_S, x_{N \setminus S}) > u_i(x)$ for all $i \in S$, and (ii) no $T \subset S, T \neq \emptyset, S$ has an ICI upon $(y_S, x_{N \setminus S})$. x is a *coalition-proof Nash equilibrium* (CPNE) if no $T \subseteq N, T \neq \emptyset$, has an ICI upon x . The reader is referred to Bernheim, Peleg, and Whinston [1987] for discussion and motivation.

As the reader may verify, the solution CPNE satisfies OPR and CONS (on closed domains). Because the set of CPNE's of a game may be a proper subset of the set of NE's, CPNE does not satisfy COCONS (see Definition 2.5). The right converse consistency concept for the CPNE solution is the following.

Definition 4.1. Let φ be a solution on a closed family of games. φ satisfies COCONS₁ if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ with $|N| \geq 2$, and every $x \in \tilde{\varphi}(G)$ (see (2.1)) the following condition holds. $x \in \varphi(G)$ if and only if there exists no $y \in \tilde{\varphi}(G)$ such that $u_i(y) > u_i(x)$ for all $i \in N$.

We are now able to prove the following result.

THEOREM 4.2. *Let Γ be a closed family of games. There is a unique solution on Γ that satisfies OPR, CONS, and COCONS₁ and it is the CPNE correspondence.*

PROOF: We shall only prove the uniqueness part. Let φ be a solution on Γ that satisfies the foregoing three axioms. We shall prove that $\varphi(G) = CPNE(G)$ for every $G \in \Gamma$ by induction on the number of players. If $G \in \Gamma$ is a one-person game then $\varphi(G) = CPNE(G)$ by OPR. If the number of players in G is $k \geq 2$, then, by the induction hypothesis, $\tilde{\varphi}(G) = \widetilde{CPNE}(G)$. Now let $x \in \varphi(G)$. By CONS $x \in \tilde{\varphi}(G)$. Also, by COCONS₁, there exists no $y \in \tilde{\varphi}(G)$ such that $u_i(y) > u_i(x)$ for all $i \in N$. Again by COCONS₁, $x \in CPNE(G)$. Thus, $CPNE(G) \supset \varphi(G)$. Similarly, $\varphi(G) \supset CPNE(G)$. Q.E.D.

We shall now show that the three axioms of Theorem 4.2 are independent. Indeed, let $\varphi(G) = \emptyset$ for each $G \in M$ (see Section 2 where M is defined). Then φ satisfies CONS and COCON₁ but not OPR. Also, the NE solution on M satisfies OPR and CONS but not COCONS₁. The following example completes the proof of the independence of the three axioms.

Example 4.3. Let $G^1 = (N^1, (A_i^1)_{i \in N}, (u_i^1)_{i \in N}) \in M$ satisfy $N^1 = \{1, 2, 3\}$ and $CPNE(G^1) = \emptyset$. Define $G^0 = (N^0, (A_i^0)_{i \in N^0}, (u_i^0)_{i \in N^0})$ in M by $N^0 = \{1, 2, 3, 4\}$, $A_i^0 = A_i^1$, $i \in N$, and $u_i^0(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = u_i^1(\alpha_1, \alpha_2, \alpha_3)$ for all $\alpha_i \in A_i^1$ and all $i \in N^1$. Denote by Γ the minimum closed family that contains G^0 . We now define a solution φ on Γ by the following rule. If $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is in Γ and $|N| \leq 3$, then $\varphi(G) = CPNE(G)$, and $\varphi(G^0) = A^0$. As the reader may verify, φ satisfies OPR and COCONS₁ but not CONS.

Kaplan [1992] has introduced a new interesting solution concept, the semi-strong Nash equilibrium. We shall now present Kaplan's definition.

Definition 4.4. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game. $x \in A$ is a *semi-strong Nash equilibrium* (SMSNE) if for every $\emptyset \neq S \subseteq N$ and every $y_S \in NE(G^{S,x})$ there exists $i \in S$ such that $u_i(x) \geq u_i(y_S, x_{N \setminus S})$.

The reader is referred to Kaplan [1992] for discussion and motivation.

Remark 4.5. If $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is a game in strategic form with continuous payoffs and compact strategy sets, then

$$(4.1) \quad CPNE(G) \supset SMSNE(G) \supset SNE(G)$$

We also notice that SMSNE satisfies OPR (if the utilities are continuous and the strategy sets are compact), and CONS. In order to characterize Kaplan's solution, we need an additional version of converse consistency.

Definition 4.6. Let Γ be a closed family of games and let φ be a solution on Γ . φ satisfies COCONS₂ if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ with $|N| \geq 2$ and for every $x \in A$ the following condition holds. If $x \in \tilde{\varphi}(G)$ and there exists no $y \in NE(G)$ such that $u_i(y) > u_i(x)$ for all $i \in N$, then $x \in \varphi(G)$.

For the next theorem we assume continuity of the payoffs and compactness of the strategy sets.

THEOREM 4.7. *Let Γ be a closed family of games. There is a unique solution on Γ that satisfies OPR, CONS, and COCONS₂ and it is the SMSNE solution. Furthermore, the foregoing three axioms are independent.*

The proof of Theorem 4.7 is left to the reader.

Remark 4.8. The solution SMSNE may be useful because of the following two reasons:

- (i) It may be non-empty when SNE is empty; (ii) In some cases, it may be easier to compute than the CPNE (see Kaplan [1992]).

5. A Characterization of Dominant Strategies

In this section we shall axiomatize the solution which assigns to every game in strategic form the (possibly empty) set of dominant strategies. We start with some definitions.

Definition 5.1. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a game. $d_i \in A_i$ is a *dominant* (DOM) strategy for player i if $u_i(d_i, a_{N \setminus \{i\}}) \geq u_i(e_i, a_{N \setminus \{i\}})$ for all $e_i \in A_i$ and $a_{N \setminus \{i\}} \in A_{N \setminus \{i\}}$. We also denote

$$(5.1) \quad DOM(G) = \{d \in A \mid d_i \text{ is a dominant strategy for all } i \in N\}.$$

The solution DOM satisfies the following two new properties.

Definition 5.2. Let Γ be a set of games and let φ be a solution on Γ . φ is *solvable* (SOL) if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ there exist sets $(B_i)_{i \in N}$, such that $B_i \subseteq A_i$ for all $i \in N$ and $\varphi(G) = \prod_{i \in N} B_i$.

Remark 5.3. Definition 5.2 is inspired by Nash [1951] (see also Luce and Raiffa [1957, Section 5.9]). However, we use solvability in a different meaning, namely as a property of solutions, whereas Nash used it as a property of games.

Clearly, DOM satisfies solvability. The next property is peculiar to DOM.

Definition 5.4. Let Γ be a closed* family of games (see Section 2), and let φ be a solution on Γ . φ has the *decomposition property* (DP) if for every $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$, $i \in N$, and a partition $(A_i^\alpha)_{\alpha \in J}$ of A_i , the following condition is satisfied. Let $G_\alpha = (N, (A_j^\alpha)_{j \in N}, (u_j)_{j \in N})$ where $A_j^\alpha = A_j$ if $j \neq i$, let $B_{-i}^\alpha = \{x_{N \setminus \{i\}} \in A_{N \setminus \{i\}} | x \in \varphi(G_\alpha)\}$, and $B_{-i} = \{x_{N \setminus \{i\}} \in A_{N \setminus \{i\}} | x \in \varphi(G)\}$. Then $B_{-i} = \bigcap_{\alpha \in J} B_{-i}^\alpha$.

DP is a strong condition. However, its interpretation is straightforward.

Remark 5.5. Let $G \in P$ (see (2.3)). Then a (pure) strategy is dominant in G if and only if it is dominant in the mixed extension of G . Furthermore, G has a dominant strategy if and only if its mixed extension has one. Hence, in studying dominant strategies, we may restrict ourselves to pure strategies. In particular, the restriction to closed* families of games in Definition 5.4 is not too restrictive.

The following theorem is the main result of this section.

THEOREM 5.6. Let Γ be a closed² family of games. There is a unique solution on Γ that satisfies OPR, DUM, SOL, and DP, and it is the DOM correspondence (see (5.1)).

PROOF: We have only to prove the uniqueness part. Let φ be a solution on Γ that satisfies the foregoing four properties. We shall prove that φ coincides with DOM by induction on the number of players. If $G \in \Gamma$ is a one-person game then $\varphi(G) = \text{DOM}(G)$ by OPR. Now let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a $(k+1)$ -person game where $k \geq 1$, let $i \in N$,

and let $x_i \in A_i$. The game $G(x_i) = (N, (A_j^*)_{j \in N}, (u_j)_{j \in N})$ where $A_j^* = A_j$ if $j \neq i$ and $A_i^* = \{x_i\}$ is in Γ because Γ is closed². By DUM

$$(5.2) \quad \varphi(G(x_i)) = \{x_i\} \times \varphi(G^{N \setminus \{i\}, x_i})$$

(The notation $G^{N \setminus \{i\}, x_i}$ is justified by the observation that $G^{N \setminus \{i\}, y}$ depends only on y_i for every $y \in A$). Now let $B_{-i} = \{x_{N \setminus \{i\}} | x \in \varphi(G)\}$, $B_{-i}^{x_i} = \{x_{N \setminus \{i\}} | x \in \varphi(G(x_i))\}$, $\bullet B_{-i}^{x_i} = \{x_{N \setminus \{i\}} | x \in \text{DOM}(G(x_i))\}$, and $\bullet B_{-i} = \{x_{N \setminus \{i\}} | x \in \text{DOM}(G)\}$. Then, by (5.2), $B_{-i}^{x_i} = \varphi(G^{N \setminus \{i\}, x_i})$ and $\bullet B_{-i}^{x_i} = \text{DOM}(G^{N \setminus \{i\}, x_i})$. Hence, by DP and the induction hypothesis

$$B_{-i} = \bigcap_{x_i \in A_i} B_{-i}^{x_i} = \bigcap_{x_i \in A_i} \varphi(G^{N \setminus \{i\}, x_i}) = \bigcap_{x_i \in A_i} \text{DOM}(G^{N \setminus \{i\}, x_i}) = \bigcap_{x_i \in A_i} \bullet B_{-i}^{x_i} = \bullet B_{-i}.$$

We may conclude now that the projection of $\varphi(G)$ on every A_j , $j \neq i$, is equal to the projection $\text{DOM}(G)$ on A_j . Similarly, we can prove that $\varphi(G)$ and $\text{DOM}(G)$ have the same projection on A_i . Using solvability we conclude that $\varphi(G) = \text{DOM}(G)$. Q.E.D.

Remark 5.7. We notice that DUM and DP imply CONS. The proof follows from the proof of Theorem 5.6.

Remark 5.8. The solution DOM has the following additional properties: CONS, PO, and IIS.

We shall now prove that the four properties of Theorem 5.6 are independent. First we notice that Example 2.15 satisfies DUM, SOL, and DP, but not OPR. We continue with the following examples.

Example 5.9. Define a solution φ on P (see (2.3)) by the following rule. Let $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \in P$. If $|N| = 1$, let

$$\varphi(G) = \{x_i \in A_i | u_i(x_i) \geq u_i(y_i) \text{ for all } y_i \in A_i\}$$

where $N = \{i\}$. If $|N| \geq 2$ let $\varphi(G) = \emptyset$. As the reader may verify, φ satisfies OPR, SOL, and DP but not DUM.

Example 5.10. Let Γ be the minimum closed² family that contains the following two-person game G_0 (in pure strategies)

	L	R
T	9, 9	6, 8
B	8, 6	7, 7

Define a solution φ on Γ by: $\varphi(G_0) = \{(T, L)\}$, and $\varphi(G) = NE(G)$ if $G \in \Gamma$, $G \neq G_0$. Then φ satisfies OPR, DUM, and SOL but not DP (because $\varphi \neq DOM$).

Remark 5.11. Example 5.10 satisfies also PO and IIS.

Example 5.12. Let Γ be the minimum closed family that contains the following two-person game G_0 (in pure strategies)

	L	R
T	0, 0	0, 0
B	0, 0	0, 0

Define a solution φ on Γ by: $\varphi(G_0) = \{(T, L), (B, R)\}$, and $\varphi(G) = NE(G)$ if $G \in \Gamma$, $G \neq G_0$. Then φ satisfies OPR, DUM, and DP but not SOL.

6. Bayesian Games

All our results may be generalized to Bayesian games. The generalization of Theorem 2.12 will be given in this section. In order to define reduced games of Bayesian games we first have to modify the definition of Bayesian games.

Definition 6.1. An *extended* Bayesian game is a system

$$G = (N, (A_i)_{i \in N}, (T_i)_{i \in N_+}, (p_i)_{i \in N}, (u_i)_{i \in N})$$

where

- (i) N is the set of *players*;
- (ii) $N_+ \supset N$ and $L = N_+ \setminus N$ is the set of *outside* players ($L = \emptyset$ is possible);
- (iii) A_i is the set of *actions* of $i \in N$;
- (iv) T_i is the finite set of possible *types* of $i \in N_+$;
- (v) $p_i : T_i \rightarrow \Delta(T_{-i})$, where $T_{-i} = \prod_{k \in N_+ \setminus \{i\}} T_k$ and $\Delta(T_{-i})$ is the set of all probability distributions on T_{-i} , represents the *beliefs* of $i \in N$; and
- (vi) $u_i : A \times T \rightarrow R$ where $T = \prod_{k \in N_+} T_k$, is the *utility function* of $i \in N$. (We recall that R is the set of real numbers and $A = \prod_{k \in N} A_k$.)

Definition 6.1 is due to Einy and Peleg [1991]. It is justified by the fact that a reduced game of a Bayesian game is an extended Bayesian game (and not an ordinary Bayesian game).

Let $G = (N, (A_i)_{i \in N}, (T_i)_{i \in N_+}, (p_i)_{i \in N}, (u_i)_{i \in N})$ be an extended Bayesian game. A *strategy* of a player $i \in N$ is a function $x_i : T_i \rightarrow A_i$. We denote by X_i the set of all strategies of player i . For $\emptyset \neq S \subseteq N$, $X_S = \prod_{i \in S} X_i$, and $X = X_N$. Let Γ be a set of extended Bayesian games. A *solution* on Γ is a function φ that assigns to each game $G = (N, (A_i)_{i \in N}, (T_i)_{i \in N_+}, (p_i)_{i \in N}, (u_i)_{i \in N}) \in \Gamma$ a subset $\varphi(G)$ of X . We shall be interested in the following solution.

Definition 6.2. Let $G = (N, (A_i)_{i \in N}, (T_i)_{i \in N_+}, (p_i)_{i \in N}, (u_i)_{i \in N})$ be an extended Bayesian game. $x \in X$ is a *Bayesian equilibrium* (BE) of G if for all $i \in N$, $t_i \in T_i$, and $a_i \in A_i$

$$\sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i((x_j(t_j))_{j \in N}, t) \geq \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(((x_j(t_j))_{j \neq i}, a_i), t).$$

We denote

$$BE(G) = \{x \in X | x \text{ is a BE of } G\}.$$

We shall now extend the formulation of the properties that characterize the NE solution.

Definition 6.3. Let φ be a solution on a set Γ of extended Bayesian games. φ satisfies one-person rationality (OPR) if for every one-person game $G = (\{i\}, A_i, (T_j)_{j \in \{i\}^+}, p_i, u_i)$ in Γ

$$\varphi(G) = \{x_i \in X_i | U_i(x_i | t_i) \geq U_i(y_i | t_i) \text{ for all } t_i \in T_i \text{ and } y_i \in X_i\}$$

where $U_i(z_i | t_i)$ for $z_i \in X_i$ and $t_i \in T_i$ is defined by

$$U_i(z_i | t_i) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} | t_i) u_i(z_i(t_i), t).$$

Now let $G = (N, (A_i)_{i \in N}, (T_i)_{i \in N^+}, (p_i)_{i \in N}, (u_i)_{i \in N})$ be an extended Bayesian game, let $\emptyset \neq S \subseteq N$, and let $x \in X$. The *reduced game of G w.r.t. S and x* is the extended Bayesian game

$$G^{S,x} = (S, (A_i)_{i \in S}, (T_i)_{i \in N^+}, (p_i)_{i \in S}, (u_i^x)_{i \in S})$$

where $(u_i^x)_{i \in S}$ are defined by the following rule.

Let $a_S \in A_S$ and let $t \in T$. Then

$$(6.1) \quad u_i^x(a_S, t) = u_i((a_S, (x_j(t_j))_{j \in N \setminus S}), t)$$

A family Γ of extended Bayesian games is *closed* if for every $G = (N, (A_i)_{i \in N}, (T_i)_{i \in N^+}, (p_i)_{i \in N}, (u_i)_{i \in N})$ in Γ , $\emptyset \neq S \subseteq N$, and $x \in X$, the reduced game $G^{S,x} \in \Gamma$.

Let Γ be a closed family of games and let φ be a solution on Γ . For a game $G = (N, (A_i)_{i \in N}, (T_i)_{i \in N^+}, (p_i)_{i \in N}, (u_i)_{i \in N})$ in Γ with $|N| \geq 2$ we denote

$$(6.2) \quad \tilde{\varphi}(G) = \{x \in X | \text{for every } S \subset N, S \neq \emptyset, N, x_S \in \varphi(G^{S,x})\}$$

We are now able to define consistency and converse consistency.

Definition 6.4. A solution φ on a closed family of extended Bayesian games Γ satisfies *consistency* (CONS), (*converse consistency* (COCONS)), if for every $G \in \Gamma$ with at least two players $\tilde{\varphi}(G) \supset \varphi(G)$ ($\tilde{\varphi}(G) \subset \varphi(G)$).

The generalization of Theorem 2.12 is now possible.

THEOREM 6.5. *Let Γ be a closed family of extended Bayesian games. There is a unique solution on Γ that satisfies OPR, CONS, and COCONS, and it is the BE correspondence.*

The proof of Theorem 6.5 is left to the reader.

Remark 6.6. The properties OPR, CONS, and COCONS refer to players and *not* to types. Hence, it is impossible to use the type-agent representation of Bayesian games (see Myerson [1991, Section 2.8]), in order to “translate” our former results on Nash equilibria to Bayesian equilibria. As far as we can see, the introduction of extended Bayesian games is essential for the generalization of our results in Sections 2–5.

7. Extensive Form Games

All the results in Sections 2–5 can be generalized to games in extensive form. We will show this for Theorem 2.12. Also, we shall provide an axiomatic characterization of the correspondence of subgame perfect equilibria on the set of games with perfect information.

Let $G = (N, K, P, U, C, p, r)$ be an extensive game. N is the set of players, K is the (finite) *game tree*, P is the *player partition*, U is the *information partition*, C is the *choice partition*, p is the *probability assignment*, and r is the *payoff function* (see van Damme [1987], Section 6.1). We shall only deal with games with perfect recall. Hence, we may restrict ourselves to behavior strategies (Kuhn [1953]). We shall denote by B_i the set of behavioral strategies of player $i \in N$. Let $i \in N$, let $b_i \in B_i$, and let $u \in U_i$ be an information set of i . Then we denote by b_{iu} the local strategy of i at u , that is, his probability distribution on the set C_u of choices at u . Also, for $\emptyset \neq S \subseteq N$, $B_S = \prod_{i \in S} B_i$, and $B = B_N$.

Let Γ be a set of games in extensive form (with perfect recall). A *solution* on Γ is a function φ that assigns to each game $G = (N, K, P, U, C, p, r) \in \Gamma$ a subset $\varphi(G)$ of B . First, we shall be interested in the solution represented by the NE correspondence (in behavior strategies). For the sake of completeness we repeat the following definition.

Definition 7.1. Let φ be a solution on a set Γ of games in extensive form. φ is *one-person rational* (OPR) if for every one-person game $G = (\{i\}, K, P, U, C, p, r)$ in Γ

$$\varphi(G) = \{x_i \in B_i \mid R_i(x_i) \geq R_i(y_i) \text{ for } y_i \in B_i\}$$

where $R_i(z_i)$, for $z_i \in B_i$, is the expected payoff to i when he plays z_i .

Reduced games of extensive form games were defined in Bernheim, Peleg, and Whinston [1987]. Here we modify the original definition in order to get a more convenient version. Let $G = (N, K, P, U, C, p, r)$ be an extensive game, let $\emptyset \neq S \subseteq N$, and let $b = (b_i)_{i \in N} \in B$. The *reduced game of G w.r.t. S and b* is the game in extensive form $G^{S,b} = (S, K^*, P^*, U^*, C^*, p^*, r^*)$ which is obtained from G in the following way. First, for every $i \in N \setminus S$ and $u \in U_i$ we decompose u into $|u|$ singletons and add them to P_0 (i.e., they belong to $P_0^* \setminus P_0$). To each $x \in u$ we now assign the probability distribution $b_{i,u}$. Thus, we eliminate P_i , $i \in N \setminus S$ and U_i , $i \in N \setminus S$ and modify P_0 . Secondly, we also replace $r = (r_i)_{i \in N}$ by $r^* = (r_i)_{i \in S}$. Furthermore, C^* is the restriction to $\cup_{i \in S} U_i$ of C . As the reader may verify, if G has perfect recall, then also $G^{S,b}$ has perfect recall.

Remark 7.2. $G^{S,b}$ may not be a game according to van Damme [1987] because of the following reason. Let $i \in N \setminus S$, $u \in U_i$ and $x \in u$. If $b_{i,u}$ is not completely mixed, then an alternative at $x \in P_0^*$ may be assigned a zero probability, a violation of van Damme's definition. However, some authors do not insist on positive probability at chance moves (see, e.g., Myerson [1991]). Moreover, Kuhn's theorem, as formulated in Myerson [1991] does not depend on it.

We now proceed to define consistency and converse consistency. A family Γ of extensive games is *closed* if for every $G = (N, K, P, U, C, p, r) \in \Gamma$, $\emptyset \neq S \subseteq N$, and $b \in B$, the reduced game $G^{S,b} \in \Gamma$. Let Γ be a closed family of games and let φ be a solution on Γ . For a game $G = (N, K, P, U, C, p, r) \in \Gamma$ with $|N| \geq 2$ we denote

$$(7.1) \quad \tilde{\varphi}(G) = \{b \in B \mid b_S \in \varphi(G^{S,b}) \text{ for every } S \subset N, S \neq \emptyset, N\}.$$

Definition 7.3. A solution φ on a closed family of extensive games Γ satisfies *consistency* (CONS), (*converse consistency* (COCONS)), if for every $G \in \Gamma$ with at least two players $\tilde{\varphi}(G) \supset \varphi(G)$ ($\tilde{\varphi}(G) \subset \varphi(G)$).

We may now generalize Theorem 2.12.

THEOREM 7.4. *Let Γ be a closed family of extensive games (with perfect recall). There is a unique solution on Γ that satisfies OPR, CONS, and COCONS and it is the NE correspondence (in behavioral strategies).*

The proof of Theorem 7.4 is left to the reader.

Let $G = (N, K, P, U, C, p, r)$ be an extensive game with perfect recall. A behavioral strategy $b \in B$ is a *subgame perfect equilibrium* (SPE) of G if for every subgame G_* of G , b_* , the restriction of b to G_* , is a NE of G_* . Let Γ_p be the set of all games with perfect information. The reader may verify that Γ_p is closed. We shall now characterize the SPE solution on Γ_p . First we need to modify OPR.

Definition 7.5. Let φ be a solution on Γ_p . φ is *perfectly OPR* (POPR) if for every one-person game $G = (\{i\}, K, P, U, C, p, r)$ in Γ

$$\varphi(G) = \{x_i \in B_i | x_i \text{ is an SPE of } G\}.$$

As the reader may easily verify POPR is stronger than OPR even on Γ_p . POPR is, simply, the principle of backward induction for one-person decision problems with perfect information (i.e., if there are chance moves, then the player knows their outcomes with certainty). We are now ready to prove the following theorem.

THEOREM 7.6. *There is a unique solution on Γ_p that satisfies POPR, CONS, and COCONS and it is the SPE correspondence.*

PROOF: SPE satisfies POPR by definition. CONS follows from our definition of reduced games of extensive games. Thus, it remains to prove that the SPE solution satisfies COCONS on Γ_p . Let $G = (N, K, P, U, C, p, r) \in \Gamma_p$ with $|N| \geq 2$ and let $x \in \widetilde{SPE}(G)$ (see (7.1)). Furthermore, let G_* be a subgame of G . Because $x \in \widetilde{SPE}(G)$ for every $i \in N$ $x_i \in SPE(G^{(i),x})$. Clearly $G_*^{(i),x}$ is a subgame of $G^{(i),x}$ for each $i \in N$. Hence x_i^* , the restriction of x_i to the moves of i in $G_*^{(i),x}$, is an NE of $G_*^{(i),x}$. By COCONS₀ (see Definition 2.6) of NE, $x^* = (x_i^*)_{i \in N}$ is an NE of G_* . Thus, SPE satisfies COCONS.

We shall now prove the uniqueness part of Theorem 7.6. Assume that φ is a solution on Γ_p that satisfies POPR, CONS, and COCONS. We will show that $\varphi = SPE$ by induction of the number of players. Let $G \in \Gamma_p$. If G is a one-person game, the $\varphi(G) = SPE(G)$ by POPR. Thus, let the number of players of G be $n \geq 2$. By the induction hypothesis $\tilde{\varphi}(G) = \widetilde{SPE}(G)$. Hence, by CONS and COCONS

$$\varphi(G) = \tilde{\varphi}(G) = \widetilde{SPE}(G) = SPE(G) \quad \text{Q.E.D.}$$

We conclude with the following example which shows that SPE does *not* satisfy CONS on the set of *all* extensive games with perfect recall.

Example 7.7. Let G be the game of Figure 1. Then $(a_1, b_1, c_1) \in SPE(G)$ (because G has no subgames). However $(a_1, b_1) \notin SPE(G^{(1,2)}, (a_1, b_1, c_1))$ (see, again, Figure 1).

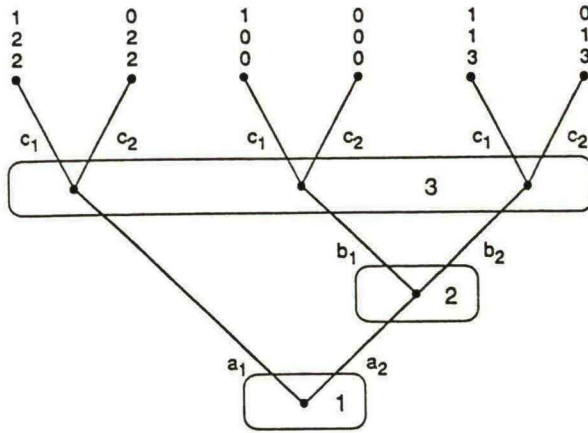
8. Concluding Remarks

First we summarize our results for games in strategic form. We considered six solutions which are ordered by inclusion in the following way

$$(8.1) \quad NE \supset CPNE \supset SMSNE \supset SNE \supset SSNE \supset DOM$$

(see Sections 2–5 for our abbreviations).

All the solutions in (8.1) satisfy the two basic axioms: one-person rationality (OPR), and consistency (CONS). Each of the first five solutions also satisfies some versions of converse consistency (COCONS). For CPNE and SMSNE we weakened COCONS by combining it with some version of restricted Pareto optimality (see Definitions 4.1 and 4.6). For SNE and SSNE we have to add full WPO and PO respectively (see Definitions 3.1 and 3.6 and Theorems 3.2 and 3.7). The axiomatization of dominant strategies required the introduction of two new strong properties: DP and SOL (see Definitions 5.2 and 5.4 and Theorem 5.6). Two new axioms, IIS and DUM (see Definitions 2.19 and 2.20), which are simple and intuitive, were introduced in Section 2 and served to obtain alternative characterization of NE and SNE (see Corollaries 2.22 and 3.9). In each of our characterizations the axioms which appear are independent.



G

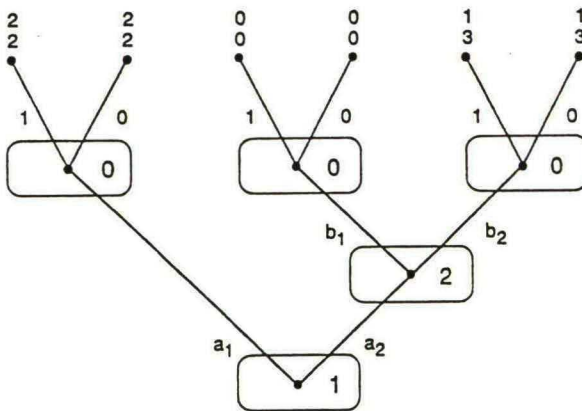
 $G(1,2),(a_1,b_1,c_1)$

Figure 1

Secondly, we remark on possible generalizations of our results. As we indicated in Sections 6 and 7, all our results may be generalized to Bayesian and extensive games. Moreover, on the class of extensive games with perfect information we axiomatized the SPE (subgame perfect equilibrium) correspondence (see Theorem 7.6).

Finally, we mention some open problems. The axiomatization of the following solutions: perfect equilibria (Selten [1975]), proper equilibria (Myerson [1978]), persistent equilibria (Kalai and Samet [1984]), and stable equilibria (Kohlberg and Mertens [1986]), have not yet been obtained. However, we notice that all these solutions do not satisfy CONS (see Example 2.4).

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